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HYDRODYNAMICS IN TECHNOLOGICAL PROCESSES

VARIATIONAL APPROACH TO FREE CONVECTION BOUNDARY-LAYER FLOW WITH SUCTION AND INJECTION

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The Gyarmati variational principle — a significant development in the field of the thermodynamics of irreversible processes — is employed to study suction and injection effects in flow and heat transfer in a free convection boundary layer over a cone. The velocity and temperature distributions inside respective boundary layers are considered as simple polynomial functions, and with the use of the perturbation procedure the variational principle is formulated. The Euler–Lagrange equations are reduced to coupled polynomial equations in terms of boundary-layer thicknesses. The skin-friction (shear-stress) and heat-transfer (Nusselt number) values with constant wall temperature are computed for various values of the suction and injection parameters and the cone-angle parameter. The comparison of the present solution with an available numerical solution shows good agreement.

Introduction. The prime objective of this study is to demonstrate the utility of a thermodynamic method in solving boundary-layer and heat-transfer problems with a reasonable accuracy. According to the boundary-layer theory, the irreversible processes of momentum and heat transfer in the flows around bodies occur mainly inside thin layers adjacent to the body surface. Therefore, it is quite appropriate to study these nonequilibrium processes by a technique based on irreversible thermodynamics. The accuracy of the present solutions is compared with that of the known solutions and is found satisfactory.

The mass-transfer effect on free convection boundary layers on surfaces of general shape has been investigated by many researchers. Eichhorn [1] found conditions for the wall temperature and transpiration rates under which similarity solutions are possible for a vertical porous plate. Sparrow and Cess [2] discussed approximate series solutions for uniform wall temperature and transpiration velocity. Merkin [3] gave an asymptotic series solution for two-dimensional bodies. Clarke [4] presented solutions for the outer region of the flow field for blowing conditions under which similar solutions to the boundary-layer equations are obtained.

The existence of a similarity solution for the axisymmetric laminar free convection flow from an isothermal vertical cone was reported by Merk and Prins [5]. Hering and Grosh [6] found that similarity solutions for the boundary-layer equations exist when the surface temperature varies as x^n. Numerical solutions of the transformed equations were presented for Prandtl number Pr = 0.7 with different temperature distributions. Later, Hering [7] extended the analysis to low Prandtl number fluids and obtained numerical solutions for liquid metals as well as for an inviscid fluid. Roy [8] extended the work of Hering and Grosh to high Prandtl number fluids. Watanabe [9] considered a nonsimilar free convection boundary-layer flow with uniform suction and injection over a vertical flat plate by employing the difference differential method. Recently, Watanabe [10] performed numerical calculations of the integral equations for various values of suction and injection parameters and cone-angle parameter with Prandtl number Pr = 0.73 by iterative numerical quadratures.

Equations of Motion. A steady, two-dimensional, laminar, free convection, boundary-layer flow over a cone with suction and injection is considered. The coordinate system (Fig. 1) is such that x is the distance from the apex along the surface of the body, with $x = 0$ being the leading edge and y the distance along the outward normal. The

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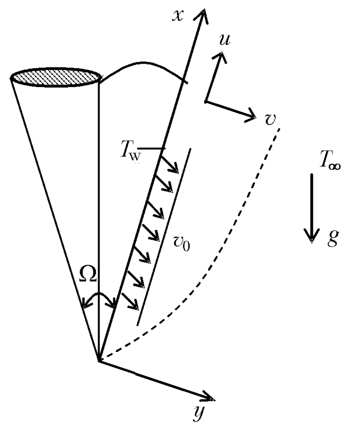

Fig. 1. Geometrical scheme of the problem.

body is held at a constant temperature T_0 higher than the ambient temperature T_∞. If we assume that $(T_0 — T_\infty)/T_\infty << 1$, viscous dissipation can be neglected and the changes in density are important only for buoyancy forces. The boundary-layer equations are

$$u_x + v_y = 0 \,, \tag{1}$$

$$uu_x + vu_y = \nu u_{yy} + gB\,(T - T_\infty)\cos(\Omega/2) \,, \tag{2}$$

$$uT_x + vT_y = \alpha T_{yy} \,, \tag{3}$$

where the subscripts signify partial derivatives. The coefficient of volume expansion B is replaced by $1/T_\infty$.

The governing equations (1)–(3) satisfy the following boundary conditions:

$$\begin{aligned} y = 0 :\quad & u = 0 \,,\quad v = v_0 \quad \text{(constant)} \,,\quad T = T_0 \,; \\ y = \infty :\quad & u = 0 \,,\quad T = T_\infty \,. \end{aligned} \tag{4}$$

Formulation of Gyarmati's Principle. Gyarmati [11, 12] proposed a genuine variational principle, "Governing Principle of Dissipative Processes" (GPDP), which is given in universal form:

$$\delta \int_V [\sigma - \Psi - \Phi]\, dV = 0 \,. \tag{5}$$

The principle (5) is valid for linear, quasi-linear, and certain types of nonlinear transport processes at any instant of time provided the following balance equations are satisfied:

$$\rho \dot{a}_i + \nabla \cdot \mathbf{J}_i = \sigma_i \quad (i = 1, 2, 3, \ldots, f) \,. \tag{6}$$

Here, $\mathbf{J}_i$ is the flux and σ_i is the source density of the ith extensive quantity a_i, and the entropy production σ can always be written in the bilinear form

$$\sigma = \sum_{i=1}^{f} \mathbf{J}_i \cdot \mathbf{X}_i \geq 0 \,, \tag{7}$$

where $\mathbf{J}_i$ and $\mathbf{X}_i$ are the fluxes and forces, respectively. According to the Onsager theory [13, 14], the fluxes are linear functions of the forces:

$$\mathbf{J}_i = \sum_{k=1}^{f} L_{ik}\mathbf{X}_k \tag{8}$$

or alternatively

$$\mathbf{X}_i = \sum_{k=1}^{f} R_{ik}\mathbf{J}_k \,. \tag{9}$$

The constants L_{ik} and R_{ik} are, respectively, the conductivities and resistances which satisfy the reciprocal relations

$$L_{ik} = L_{ki} \,, \quad R_{ik} = R_{ki} \tag{10}$$

and the matrices L_{ik} and R_{ik} are the mutual reciprocals:

$$\sum_{m=1}^{f} L_{im}R_{mk} = \sum_{m=1}^{f} L_{mk}R_{im} = \delta_{ik} \,, \tag{11}$$

where δ_{ik} is the Kronecker delta.

The local dissipation potentials Ψ and Φ are defined as

$$\Psi\,(\mathbf{X}, \mathbf{X}) = (1/2) \sum_{i,\,k=1}^{f} L_{ik}\mathbf{X}_i \cdot \mathbf{X}_k \geq 0 \,, \tag{12}$$

$$\Phi\,(\mathbf{J}, \mathbf{J}) = (1/2) \sum_{i,k=1}^{f} R_{ik}\mathbf{J}_i \cdot \mathbf{J}_k \geq 0 \,. \tag{13}$$

In the case of transport processes, $\mathbf{X}_i$ can be generated as a gradient of a certain variable Γ:

$$\mathbf{X}_i = \nabla\Gamma \,, \tag{14}$$

so that with the aid of (7), (10), (12), and (13) the principle (5) assumes the form

$$\delta \int_V \left[\sum_{i=1}^{f} \mathbf{J}_i \cdot \nabla\Gamma_i - (1/2) \sum_{i,k=1}^{f} L_{ik}\nabla\Gamma_i \cdot \nabla\Gamma_k - (1/2) \sum_{i,k=1}^{f} R_{ik}\mathbf{J}_i \cdot \mathbf{J}_k \right] dV = 0 \,. \tag{15}$$

The principle (5) is also involved in the energy picture [11] as

$$\delta \int_V [T\sigma - \Psi^* - \Phi^*]\, dV = 0 \,. \tag{16}$$

Here $T\sigma$ is the energy dissipation and the dissipation potentials Ψ^* and Φ^* are given as

$$\Psi^* = T\Psi \,, \quad \Phi^* = T\Phi \,. \tag{17}$$

It is found that the GPDP in the energy picture (16) is always advantageous for dealing with thermohydrodynamical systems. Vincze [15] applied Gyarmati's principle (16) and obtained an explicit and practically applicable theory for thermohydrodynamical systems. Antony Raj and Chandrasekar [16–18] applied this variational principle to flow and heat transfer in boundary-layer flows.

In order to formulate the principle (16) for the present problem, we write the governing equations of motion in the balance form:

$$\nabla\cdot\mathbf{V} = 0 \quad (\mathbf{V} = \mathbf{i}u + \mathbf{j}v)\,, \tag{18}$$

$$\rho\,(\mathbf{V}\cdot\nabla)\,\mathbf{V} + \nabla\cdot\mathbf{P} = gB\rho\,(T - T_\infty)\,\mathbf{i}\cos(\Omega/2)\,, \tag{19}$$

$$\rho C_p\,(\mathbf{V}\cdot\nabla)\,T + \nabla\cdot\mathbf{J}_q = 0\,, \tag{20}$$

where **i** and **j** are the unit vectors in the x and y directions.

The pressure tensor **P** is given by [12]

$$\mathbf{P} = p\boldsymbol{\delta} + \mathbf{P}^{\mathrm{vs}}\,, \tag{21}$$

where $\mathbf{P}^{\mathrm{vs}}$ is the symmetrical part of **P**, whose trace is zero; $\boldsymbol{\delta}$ is the unit tensor. The constitutive equations for the present system are

$$P_{12} = -L_{ss}\,(\partial u/\partial y)\,, \quad J_q = -L_{\lambda\lambda}\,(\partial \ln T/\partial y)\,, \tag{22}$$

where $L_{\mathrm{ss}} = \mu$ and $L_{\lambda\lambda} = \lambda T$. It is well known that ln T, rather than T, is the proper state variable for the energy picture.

The energy dissipation is written as

$$T\sigma = -P_{12}\,(\partial u/\partial y) - J_q\,(\partial \ln T/\partial y) \tag{23}$$

and the dissipation potential functions Ψ^* and Φ^* take the form

$$\Psi^* = (1/2)\,[L_{\lambda\lambda}\,(\partial \ln T/\partial y)^2 + L_{ss}\,(\partial u/\partial y)^2]\,,$$
$$\Phi^* = (1/2)\,[R_{\lambda\lambda}J_q^2 + R_{ss}P_{12}^2]\,. \tag{24}$$

Using Eqs. (23) and (24), we write the principle (16) as

$$\delta\int_0^l\int_0^\infty [-J_q\,(\partial \ln T/\partial y) - P_{12}\,(\partial u/\partial y) - (L_{\lambda\lambda}/2)\,(\partial \ln T/\partial y)^2$$
$$-\,(L_{ss}/2)\,(\partial u/\partial y)^2 - (R_{\lambda\lambda}/2)\,J_q^2 - (R_{ss}/2)\,P_{12}^2]\,dydx = 0\,, \tag{25}$$

where l is the representative length of the surface.

Method of Solution. To start the thermodynamic analysis, we select the trial functions for velocity and temperature inside respective boundary layers as the following simple polynomials:

$$u = u_1\,(y/d_1)\,(1 - y/d_1)^2 \quad \text{as} \quad y < d_1\,, \quad u = 0 \quad \text{as} \quad y \ge d_1\,;$$
$$(T - T_\infty)/(T_0 - T_\infty) = (1 - y/d_2)^2 \quad \text{as} \quad y < d_2\,, \quad T = T_\infty \quad \text{as} \quad y \ge d_2\,, \tag{26}$$

where

$$u_1 = gB\,(T_0 - T_\infty)\, d_1^2/(4\nu) \tag{27}$$

is the function of x with the dimensions of velocity which is determined from Eqs. (1)–(3) and d_1 and d_2 are the velocity and temperature boundary-layer thicknesses which are to be determined from the thermodynamic analysis. The trial functions (26) satisfy the following conditions:

$$\begin{gathered} y = 0: \;\; u = 0\,, \;\; v = v_0\,, \;\; T = T_0\,, \;\; T_y = 0\,, \;\; u_{yy} = -\,gB\,(T_0 - T_\infty)/\nu\,; \\ y = d_1: \;\; u = 0\,, \;\; u_y = 0\,; \\ y = d_2: \;\; T = T_\infty\,, \;\; T_y = 0\,. \end{gathered} \tag{28}$$

Before formulating the variational principle, it is necessary to determine the expressions for the fluxes P_{12} and J_q. With the substitution of trial functions (26) in the momentum and thermal balance equations (18)–(20) and on direct integration using smooth-fit conditions (28), we obtain the following expressions for momentum and energy fluxes P_{12} and J_q, respectively:

$$\begin{gathered} -\,P_{12}/L_{\mathrm{ss}} = (1/\nu)\Big\{[gB\,(T_0 - T_\infty)/(4\nu)]^2\,(-\,1/21 + y^3/(6d_1^3) - y^5/(4d_1^5) + y^6/(6d_1^6) \\ -\,y^7/(28d_1^7))\, d_1^4 d_1' + [gB\,(T_0 - T_\infty)/(4\nu)]\,(y/d_1 - 2y^2/(3d_1^2) \\ +\,y^3/d_1^3)\, d_1^2 v_0 + [gB\,(T_0 - T_\infty)]\,(d_2/3 - y + y^2/d_2 - y^3/(3d_2^2))\cos(\Omega/2)\Big\}\,, \end{gathered} \tag{29}$$

$$\begin{gathered} -\,J_q/L_{\lambda\lambda} = (1/\alpha)\Big\{[gB\,(T_0 - T_\infty)/(4\nu)]\,[d_1^2 d_2\,(-\,d_2/(6d_1) + d_2^2/(5d_1^2) - d_2^3/(15d_1^3) \\ +\,2y^3/(3d_1d_2^2) - y^4/(d_1^2d_2^2) - y^4/(2d_1d_2^3) + 2y^5/(5d_1^3d_2^2) + 4y^5/(5d_1^2d_2^3) \\ -\,y^6/(3d_1^3d_2^3)) + d_1^2 d_1'\,(-\,d_2^2/(12d_1^2) + d_2^4/(60d_2^4) + y^3/(3d_1^2d_2) \\ -\,y^4/(3d_1^2d_2) - y^4\,(4d_1^2d_2^2) - y^5/(10d_1^4d_2) + y^6/(12d_1^4d_2^2))] + v_0\,(T_0 - T_\infty)\,[1 - 2y/d_2 + y^2/d_2^2]\Big\}\,. \end{gathered} \tag{30}$$

Using velocity and temperature trial functions (26) and Eq. (27) in the mass-conservation equation (1), we obtain the transverse velocity component v as follows:

$$\begin{gathered} v = [gB\,(T_0 - T_\infty)/(4\nu)]\,[(y^2/(2d_1^2) - 4y^3/(3d_1^3) + 3y^4/(4d_1^4))\, d_1^2 d_1' \\ +\,(-\,y^2/d_1 + 4y^3/(3d_1^2) - 2y^4/(4d_1^3))\, d_1 d_1'] + v_0\,. \end{gathered} \tag{31}$$

In the above three equations, (29)–(31), the prime signifies partial differentiation with respect to x.

With the use of the expressions for P_{12} and J_q along with Eqs. (26) and (27), the variational principle (25) is formulated. After carrying out the integration with respect to y, this principle is obtained in the following form:

$$\delta \int_0^l L\,(d_1, d_2, d_1', d_2')\, dx = 0\,, \tag{32}$$

where L is the Lagrangian density.

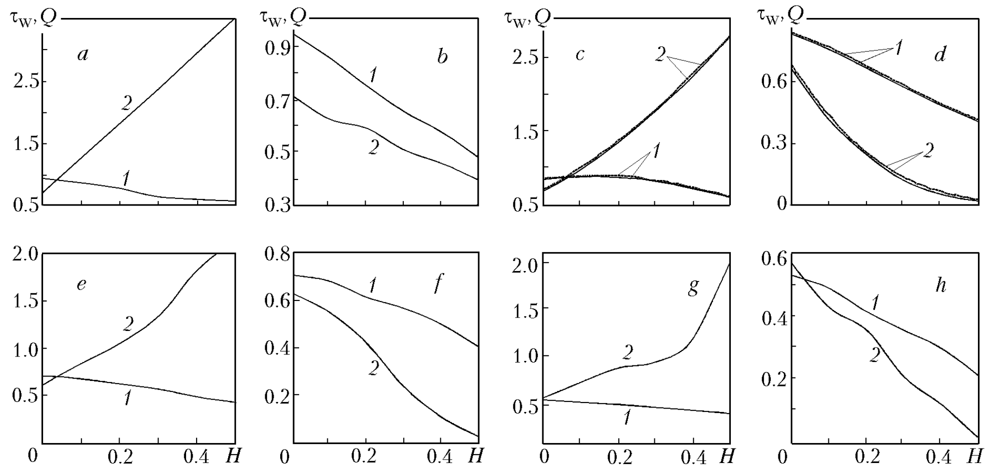

Fig. 2. Skin friction (curve 1) and heat transfer (curve 2) as functions of H for suction (a, c, e, and g) and injection (b, d, f, and h) at Pr = 0.73 and different values of m: $m = 0$ (a and b); 0.1156 (c and d); 0.24503 (e and f); 0.4241 (g and h).

The boundary-layer thicknesses d_1 and d_2 are the independent parameters to be varied, and the Euler–Lagrange equations corresponding to these variational parameters are

$$(d/dx)\,(\partial L/\partial d'_{1,2}) - (\partial L/\partial d_{1,2}) = 0\,. \tag{33}$$

Equations (33) are nonlinear ordinary differential equations of the second order in terms of d_1 and d_2. Although these equations can be solved directly with the aid of a numerical method, we can easily obtain a simple and straightforward solution employing the following transformations in the variational principle (32):

$$d_{1,2} = d^*_{1,2}\,[4\nu^2 x/gB\,(T_0 - T_\infty)]^{1/4}\,. \tag{34}$$

The Euler–Lagrange equations of the transformed principle assume the simple form

$$(\partial L/\partial d^*_{1,2}) = 0\,. \tag{35}$$

Equations (35) constitute the coupled polynomial equations for dimensionless boundary-layer thicknesses d_1^* and d_2^*, and the coefficients of these equations depend on the following independent dimensionless parameters: Prandtl number Pr, cone angle parameter m, and suction/injection speed H which is given as

$$H = [(m+3)/6]^{1/2}\,(v_0/4\nu)\,[4\nu^2 x/(gB\,(T_0 - T_\infty))]^{1/4}\,. \tag{36}$$

Suction and injection correspond to $H < 0$ and $H > 0$, respectively. The problem can be solved for any value of the Prandtl number. On the basis of the present variational technique, the nonlinear partial differential equations describing the boundary-layer flow are transformed into simple polynomial equations which are of much practical use to any practicing engineer.

After obtaining the solution for d_1^* and d_2^* at the given values of the parameters mentioned, we can calculate the skin-friction and heat-transfer values with the aid of the following expressions:

$$\tau_w = (-P_{12}/L_{ss})_{y=0}\,[v_0/(gB\,(T_0 - T_\infty))]\,, \quad Q = (J_q/L_{\lambda\lambda})_{y=0}\,[\nu/(v_0\,(T_0 - T_\infty))]\,.$$

The present analysis is carried out for Pr = 0.73, although Eqs. (35) are valid for any combination of Pr, suction/injection parameter H, and cone-angle parameter m.

Results and Discussion. The main results of engineering interest are the skin-friction (τ_w) and heat-transfer (Q) values, which is why these two important characteristics are analyzed here. The values of skin friction and heat transfer for the boundary-layer flow over a cone in the case of constant surface temperature and Pr = 0.73 at various values of H and m are computed and presented graphically in Fig. 2. This figure shows the skin friction and heat transfer as functions of the suction/injection speed H for different cone-angle parameter m. Figures 2a, b give the skin friction and heat transfer for m = 0 ($\Omega = 0^o$). From Fig. 2b it is seen that the skin friction and heat transfer decrease as injection increases, but when suction increases the skin friction decreases and heat transfer increases (Fig. 2a). These tendencies are common for all values of suction and injection parameter H and all cone-angle parameters m at the given Pr. Figures 2c—h present the skin-friction and heat-transfer values for m = 0.1156 ($\Omega = 60^o$), m = 0.2450 ($\Omega = 90^o$), and m = 0.4241 ($\Omega = 120^o$), respectively. From the figures it follows that when the cone-angle parameter m increases, the skin friction and heat transfer decrease.

When a mathematical technique is applied to a problem, it is conventional to compare the results obtained with the available ones in order to establish the accuracy of the analysis made. Accordingly, the skin-friction and heat-transfer values are compared with the numerical solutions of Watanabe [10] (Figs. 2c, d). The agreement of the results is excellent, and the accuracy of the method is remarkable for engineering applications. It could also be noted that the order of accuracy remains the same for any combinations of Pr, H, and m.

With the aid of Eqs. (29) and (30) we can calculate the velocity and temperature distributions. When suction increases, these distributions approach the wall, and the velocity of the fluid increases on the wall surface.

Thus, the paper presents an analytical solution for free convection flow with the effects of uniform suction and injection over a cone. The governing partial differential equations are reduced to coupled polynomial equations, whose coefficients are functions of the independent parameters Pr, H, and m. The great advantage of the present technique is that the results are obtained with a remarkable accuracy and the amount of calculations is certainly less than that in numerical procedures. Hence, practicing engineers and scientists can employ this unique approximate technique based on sound physical reasoning as a powerful tool for solving boundary-layer and heat-transfer problems.

NOTATION

B, coefficient of thermal expansion; C_p, specific heat; d_1 and d_2, hydrodynamical and thermal boundary-layer thicknesses; d_1^*, d_2^*, dimensionless boundary-layer thicknesses; g, acceleration of gravity; H, suction and injection parameter; J and J_q, flux and thermal flux; L, Lagrangian; L_{ik} and R_{ik}, conductivities and resistances; L_{ss}, $L_{\lambda\lambda}$, conductivities; p, hydrostatic pressure; P_{12}, momentum flux; **P**, pressure tensor; Pr, Prandtl number; Q, heat transfer; T, fluid temperature; T_0, plate temperature; T_∞, temperature of ambient fluid; u and v, velocity components in the x and y directions; v_0, suction and injection velocity; V, total volume; **V**, velocity vector; x and y, coordinates along the plate and normal to it; **X**, force; α, thermal diffusivity; δ, symbol of variation; λ, thermal conductivity; μ, viscosity; ν, kinematic viscosity; ρ, density; σ, entropy production; τ_w, dimensionless skin friction; Ψ, Φ, local dissipation potentials; Ψ^*, Φ^*, local dissipation potentials in energy picture; Ω, cone angle.

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